APPROXIMATE SOLUTION TO PROBLEMS IN THE BOUNDARY-LAYER THEORY APPLIED TO WEAK POLYMER SOLUTIONS WITH RIGID ELLIPSOIDAL MACROMOLECULES

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Integral relations are derived for a plane laminar boundary layer of weak polymer solutions with rigid ellipsoidal macromolecules. The universal equation is written out for generalizably-similar solutions to the problem.

For approximate solutions to problems in the boundary-layer theory for Newtonian fluids, one generally uses integral relations. For a boundary layer of a weak polymer solution one can, obviously, also construct approximate solutions on the basis of integral relations applied to such media.

We will consider a steady two-dimensional flow of a weak polymer solution the macromolecules in which can be simulated hydrodynamically by rigid ellipsoids of revolution near a solid surface, in conventional boundary-layer coordinates. As the rheological equation of state for the given system will serve the one which has been derived for such media in [1] on the structural-continuum basis:

$$T_{ij} = \left(-p + \frac{1}{3}\mu_1\right)\delta_{ij} + 2\mu d_{ij} + \mu_1 \langle n_i n_j \rangle + \mu_2 d_{km} \langle n_k n_m n_i n_j \rangle + 2\mu_3 (d_{kj} \langle n_k n_i \rangle + d_{ik} \langle n_k n_j \rangle),$$

where n_i are the components of the unit orientation vector, which coincides with the rotational axis of an ellipsoidal particle, and $\langle\rangle$ is the symbol for averaging with the distribution function of the orientation angles of rotational axes [2, 3], which characterizes the orientations of polymer macromolecules in a solution due to hydrodynamic forces and due to rotational Brownian motion.

The equations of a boundary layer in such media, derived as the zeroth approximation in the general asymptotic solution of the flow equation $\rho\dot{v}_i = T_{ij,j}$ and the continuity equation $d_{ii} = 0$ for large values of the Reynolds number, are [4]:

$$\begin{gathered} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = V\frac{dV}{dx} + \frac{\partial^2 u}{\partial y^2}(\nu + \nu_2 \langle n_x^2 n_y^2 \rangle + \nu_3 \langle n_x^2 + n_y^2 \rangle) \\ + \frac{\partial u}{\partial y}\left(\nu_2 \frac{\partial}{\partial y}\langle n_x^2 n_y^2 \rangle + \nu_3 \frac{\partial}{\partial y}\langle n_x^2 + n_y^2 \rangle\right) + \nu_1 Dr \frac{\partial}{\partial y}\langle n_x n_y \rangle, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \end{gathered} \tag{1}$$

where u and v denote respectively the longitudinal and the transverse component of velocity, V denotes the velocity of the mainstream V; $\langle n_x^2 + n_y^2\rangle$, $\langle n_x^2 n_y^2\rangle$, and $\langle n_x n_y\rangle$ are certain functions of $\sigma = (\partial u/\partial y)/Dr$ [1, 4] with D_r denoting the rotational diffusivity. The boundary conditions for Eqs. (1) are the same as those used in the boundary-layer theory for Newtonian fluids.

In order to derive the integral relations, we follow the procedure in [5], i.e., we multiply the first equation in (1) by $(V-u)^k$ (k = 0, 1, 2, ...) and integrate across the boundary layer from y = 0 to y = δ or y = ∞. After a few transformations, we have then

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$$\frac{d\delta^{**}_{k+1}}{dx} + \frac{V'}{V}\left[(k+2)\,\delta^{**}_{k+1} + (k+1)\,\delta^{*}_{k+1}\right]$$

$$= -(k+1)\frac{\nu}{V}\int_0^{\infty,\delta}\left(1-\frac{u}{V}\right)^k\frac{\partial}{\partial y}\left[\left(1+\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle\right.\right.$$

$$\left.\left.+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)\frac{\partial}{\partial y}\left(\frac{u}{V}\right)+\frac{\nu_1 Dr}{\nu V}\langle n_x n_y\rangle\right]dy, \qquad (2)$$

where

$$\delta_k^* = \int_0^{\infty,\delta}\left(1-\frac{u}{V}\right)^k dy,\quad \delta_k^{**} = \int_0^{\infty,\delta}\frac{u}{V}\left(1-\frac{u}{V}\right)^k dy.$$

For the second set of integral relations [6], we multiply the first equation in (1) by y^k (k = 0, 1, 2, 3, ...) and integrate across the boundary layer over the same limits as before:

$$\frac{d}{dx}\int_0^{\infty,\delta} u(V-u)\,dy + \frac{dV}{dx}\int_0^{\infty,\delta}(V-u)\,dy$$

$$= \nu\left[\frac{\partial u}{\partial y}\left(1+\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)+\frac{\nu_1 Dr}{\nu}\langle n_x n_y\rangle\right]_{y=0},$$

$$k=0,$$

$$\frac{d}{dx}\int_0^{\infty,\delta} y^k u(V-u)\,dy - k\int_0^{\infty,\delta} y^{k-1}V(V-u)\,dy + \frac{dV}{dx}\int_0^{\infty,\delta} y^k(V-u)\,dy$$

$$= \nu k\int_0^{\infty,\delta} y^{k-1}\left[\frac{\partial u}{\partial y}\left(1+\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)\right.$$

$$\left.+\frac{\nu_1 Dr}{\nu}\langle n_x n_y\rangle\right]dy,\quad k=1,\ 2,\ 3\ \ldots \qquad (3)$$

When k = 0, both sets of integral relations (2) and (3) yield the momentum equation

$$\frac{d\delta^{**}}{dx} + \frac{V'}{V}(2+H) = \frac{\nu}{V^2}\left[\frac{\partial u}{\partial y}\left(1+\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)+\frac{\nu_1 Dr}{\nu}\langle n_x n_y\rangle\right]_{y=0}, \qquad (4)$$

where

$$H = \frac{\delta^*}{\delta^{**}}.$$

In order to arrive at an approximate solution to problems in the boundary-layer theory for weak polymer solutions on the basis of integral relations, it is necessary, as in the case of Newtonian fluids, to integrate the longitudinal velocity over a set of profiles with one or several parameters which satisfy both the boundary conditions and a certain number of contour constraints. For the parameters in the set of profiles we will select the contour constraint derived from the first equation in (1) at y = 0:

$$VV' + \nu\frac{\partial^2 u}{\partial y^2}\left(1+\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)$$

$$+\nu\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)+\nu_1 Dr\frac{\partial}{\partial y}\langle n_x n_y\rangle = 0. \qquad (5)$$

Changing here to dimensionless variables $\bar{u} = u/V$, $\bar{y} = y/\delta^{**}$, we obtain

$$\frac{\partial^2\bar{u}}{\partial\bar{y}^2}\left(1+\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)+\frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial}{\partial\bar{y}}\left(\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle\right.$$

$$\left.+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)+\frac{\nu_1}{\nu}\frac{\delta^{**}Dr}{V}\frac{\partial}{\partial\bar{y}}\langle n_x n_y\rangle+\frac{V'\delta^{**2}}{\nu} = 0. \qquad (6)$$

Here $\langle n_x^2 n_y^2\rangle_{y=0}$, $\langle n_x^2 + n_y^2\rangle_{y=0}$, and $\langle n_x n_y\rangle_{y=0}$ are functions of

$$\left[\frac{\partial u}{\partial y}\Big/ Dr\right]_{y=0} = \frac{V}{\delta^{**}Dr}\left[\frac{\partial \bar{u}}{\partial \bar{y}}\right]_{y=0}.$$

It follows from (6) that the set of profiles of longitudinal velocity in a boundary layer in a weak polymer solution must depend on two parameters: $f_1 = V'\delta^{**2}/\nu$ and $\lambda = V/\delta^{**}Dr$: the first one a geometrical parameter also used in the boundary-layer theory for Newtonian fluids, and the second one a dimensionless parameter which characterizes the relaxation properties of macromolecules during flow.

Thus, one-parameter methods of calculating the characteristics of a boundary layer are not applicable to gradiental flow of weak polymer solutions; a two-parameter method requires, in addition to the integral momentum equation (4), one more equation taken from systems (2)-(3) and thus involves unwieldy calculations even in the case of Newtonian fluids [7].

The form factor is $f_1 = 0$ for a longitudinal flow around a plate (V = const) and an approximate solution can be found with only a single integral relation. Our problem has been analyzed in [8]. Here we will show the results obtained for a specific case. The momentum thickness δ^{**} for an aqueous solution of rigid ellipsoidal macromolecules at the edge of a plate, with a volume concentration $\alpha = 0.01$ and with $a/b = 10$ and $r = \sqrt[3]{ab^2} = 10^{-7}$ m, $L = 0.1$ m, at $T = 300$°K and $V = 0.1$ m/sec is 6.7% larger than δ^{**} for the solvent alone [8].

Evidently, then, the Loitsyanskii method based on finding the generalizably-similar solutions [9] is the most preferable for approximately calculating the characteristics of a boundary layer in weak polymer solutions.

Let us now write the universal equation for weak polymer solutions. The approximate solution to system (1) will be sought in the form

$$\frac{u}{V} = \varphi\left(\frac{y}{\delta^{**}}, f_1, f_2, \ldots, \lambda\right),$$

where

$$f_k = V^{k-1}V^{(k)}z^{**k}, \quad k = 1, 2, 3, \ldots,$$

$$\lambda = \frac{V}{\delta^{**}Dr}, \quad z^{**} = \frac{\delta^{**}}{\nu}.$$

In the boundary-layer equations (1) we change to the flow function $\psi(x, y)$:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = V\frac{dV}{dx} + \frac{\partial}{\partial y}\left[\frac{\partial^2\psi}{\partial y^2}(\nu + \nu_2\langle n_x^2 n_y^2\rangle + \nu_3\langle n_x^2 + n_y^2\rangle) + \nu_1 Dr\langle n_x n_y\rangle\right]. \quad (7)$$

The boundary conditions for the outer problem of hydromechanics will be

$$\psi = \frac{\partial\psi}{\partial y} = 0 \quad \text{at} \quad y = 0, \quad \frac{\partial\psi}{\partial y} \to V(x) \quad \text{at} \quad y \to \infty,$$
$$\delta^{**} = \delta_0^{**} \quad \text{at} \quad x = x_0. \quad (8)$$

Changing in (7) and (8) to independent variables f_k ($k = 1, 2, \ldots$) and λ, we will seek the solution in the form

$$\psi = \int_0^y u\,dy = V\delta^{**}\int_0^{\frac{y}{\delta^{**}}}\varphi\left(\frac{y}{\delta^{**}}, f_1, f_2, \ldots, \lambda\right)d\left(\frac{y}{\delta^{**}}\right) = \frac{V\delta^{**}}{B}\Phi(\xi, f_1, f_2, \ldots, \lambda),$$

where $\xi = By/\delta^{**}$ and B is a normalizing constant which will be determined later on.

Thus, for determining $\Phi(\xi, f_1, f_2, \ldots, \lambda)$ we have a universal equation independent of the velocity distribution in the mainstream $V(x)$:

$$\left(1+\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)\frac{\partial^3\Phi}{\partial\xi^3}+\frac{F+2f_1}{2B^2}\Phi\frac{\partial^2\Phi}{\partial\xi^2}$$

$$+\frac{f_1}{B}\left[1-\left(\frac{\partial\Phi}{\partial\xi}\right)^2\right]+\frac{\partial^2\Phi}{\partial\xi^2}\frac{\partial}{\partial\xi}\left[1+\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right]+\frac{\nu_1}{\nu B}\frac{1}{\lambda}\frac{\partial}{\partial\xi}\langle n_x n_y\rangle$$

$$=\frac{1}{B^2}\sum_{k=1}^{\infty}\theta_k\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\xi\partial f_k}-\frac{\partial\Phi}{\partial f_k}\frac{\partial^2\Phi}{\partial\xi^2}\right)+\frac{G}{B}\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\xi\partial\lambda}-\frac{\partial\Phi}{\partial\lambda}\frac{\partial^2\Phi}{\partial\xi^2}\right) \qquad (9)$$

with the boundary conditions

$$\Phi=\frac{\partial\Phi}{\partial\xi}=0 \quad \text{at} \quad \xi=0, \quad \frac{\partial\Phi}{\partial\xi}\to 1 \quad \text{at} \quad \xi\to\infty,$$
$$\Phi=\Phi_0(\xi) \quad \text{at} \quad f_1=f_2=\ldots=\lambda=0, \qquad (10)$$

where

$$F=2\left[\zeta\left(1+\frac{\nu_2}{\nu}\langle n_x^2 n_y^2\rangle+\frac{\nu_3}{\nu}\langle n_x^2+n_y^2\rangle\right)-(2+H)f_1+\frac{\nu_1}{\nu}\langle n_x n_y\rangle\right]_{y=0},$$
$$\zeta=\left[\frac{\partial(u/V)}{\partial(y/\delta^{**})}\right]_{y=0},$$
$$\theta_k=[(k-1)f_1+kF]f_1+f_{k+1}, \quad G=\lambda\left(\frac{F}{2}-f_1\right).$$
$$k=1,2,\ldots$$

When $f_1 = f_2 = \ldots = \lambda = 0$ and $2B^2 = F$, then Eq. (9) and the boundary conditions (10) coincide with the the problem of a longitudinal stream of Newtonian fluid around a plate and, therefore, the Blasius solution [10] yields $B = 0.47$.

The case $\lambda \to 0$ corresponds to the transition from a weak polymer solution with rigid ellipsoidal macromolecules to a Newtonian fluid; according to Eq. (9), the obtained universal equation is in this case identical to the universal equation for a Newtonian fluid.

The universal equation (9) in the two-parameter approximation $\psi = (V\delta^{**}/B)\Phi(\xi, f_1, \lambda)$ can be integrated with the aid of a digital computer.

The solution of specific problems reduces then to an integration of the ordinary differential equation

$$\frac{dz^{**}}{dx}=\frac{F}{V}$$

with the boundary condition

$$z^{**}=z_0^{**}, \qquad x=x_0.$$

Along with the approximate methods, however, one may also use numerical methods for a direct solution of the boundary-layer equations (1) in the case of weak polymer solutions.

NOTATION

T_{ij}	are the components of the stress tensor;
d_{ij}	are the components of the strain rate tensor;
p	is the pressure;
μ, μ_1, μ_2, μ_3	are the rheological constants defined in [1];
δ_{ij}	is the Kronecker delta;
v_i	is the thickness of a boundary layer;
ρ	is the density of a polymer solution;
δ	is the thickness of a boundary layer;
α	is the volume concentration of macromolecules;
$2a$	is the rotational axis of a macromolecule;
$2b$	is the equatorial diameter of a macromolecule;
L	is the length of a plate.

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